

THE NUMBER OF POINTS IN A MATROID WITH NO n -POINT LINE AS A MINOR

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ABSTRACT. For any positive integer l we prove that if M is a simple matroid with no $(l+2)$ -point line as a minor and with sufficiently large rank, then $|E(M)| \leq \frac{q^{r(M)}-1}{q-1}$, where q is the largest prime power less than or equal to l . Equality is attained by projective geometries over $\text{GF}(q)$.

1. INTRODUCTION

Kung [5] proved the following theorem.

Theorem 1.1. *For any integer $l \geq 2$, if M is a simple matroid with no $U_{2,l+2}$ -minor, then $|E(M)| \leq \frac{l^{r(M)}-1}{l-1}$.*

The above bound is tight in the case that l is a prime power and M is a projective geometry. In fact, among matroids of rank at least 4, projective geometries are the only matroids that attain the bound; see [5]. Therefore, the bound is not tight when l is not a prime power. We prove the following bound that was conjectured by Kung [5,4].

Theorem 1.2. *Let $l \geq 2$ be a positive integer and let q be the largest prime power less than or equal to l . If M is a simple matroid with no $U_{2,l+2}$ -minor and with sufficiently large rank, then $|E(M)| \leq \frac{q^{r(M)}-1}{q-1}$.*

The case where $l = 6$ was resolved by Bonin and Kung in [2].

We will also prove that the only matroids of large rank that attain the bound in Theorem 1.2 are the projective geometries over $\text{GF}(q)$; see Corollary 4.2.

A matroid M is *round* if $E(M)$ cannot be partitioned into two sets of rank less than $r(M)$. We prove Theorem 1.2 by reducing it to the following result.

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Theorem 1.3. *For each prime power q , there exists a positive integer n such that, if M is a round matroid with a $PG(n-1, q)$ -minor but no U_{2,q^2+1} -minor, then $\epsilon(M) \leq \frac{q^{r(M)}-1}{q-1}$.*

For any integer $l \geq 2$, there is an integer k such that $2^{k-1} < l \leq 2^k$. Therefore, if q is the largest prime power less than or equal to l , then $l < 2q$. So, to prove Theorem 1.2, it would suffice to prove the weaker version of Theorem 1.3 where U_{2,q^2+1} is replaced by $U_{2,2q+1}$. With this in mind, we find the stronger version somewhat surprising.

We further reduce Theorem 1.3 to the following result.

Theorem 1.4. *For each prime power q there exists an integer n such that, if M is a round matroid that contains a $U_{2,q+2}$ -restriction and a $PG(n-1, q)$ -minor, then M contains a U_{2,q^2+1} -minor.*

The following conjecture, if true, would imply all of the results above.

Conjecture 1.5. *For each prime power q , there exists a positive integer n such that, if M is a round matroid with a $PG(n-1, q)$ -minor but no U_{2,q^2+1} -minor, then M is $GF(q)$ -representable.*

The conjecture may hold with $n = 3$ for all q . Moreover, the conjecture may also hold when “round” is replaced by “vertically 4-connected”.

2. PRELIMINARIES

We assume that the reader is familiar with matroid theory; we use the notation and terminology of Oxley [6]. A rank-1 flat in a matroid is referred to as a *point* and a rank-2 flat is a *line*. A line is *long* if it has at least 3 points. The number of points in M is denoted $\epsilon(M)$.

Let M be a matroid and let $A, B \subseteq E(M)$. We define $\sqcap_M(A, B) = r_M(A) + r_M(B) - r_M(A \cup B)$; this is the *local connectivity* between A and B . This definition is motivated by geometry. Suppose that M is a restriction of $PG(n-1, q)$ and let F_A and F_B be the flats of $PG(n-1, q)$ that are spanned by A and B respectively. Then $F_A \cap F_B$ has rank $\sqcap_M(A, B)$. We say that two sets $A, B \subseteq E(M)$ are *skew* if $\sqcap_M(A, B) = 0$.

We let $\mathcal{U}(l)$ denote the class of matroids with no $U_{2,l+2}$ -minor. Our proof of Theorem 1.2 relies heavily on the following result of Geelen and Kabell [3, Theorem 2.1].

Theorem 2.1. *There is an integer-valued function $\alpha(l, q, n)$ such that, for any positive integers l, q, n with $l \geq q \geq 2$, if $M \in \mathcal{U}(l)$ is a matroid with $\epsilon(M) \geq \alpha(l, q, n)q^{r(M)}$, then M contains a $PG(n-1, q')$ -minor for some prime-power $q' > q$.*

The following result is an important special case of Theorem 1.4.

Lemma 2.2. *If M is a round matroid that contains a $U_{2,q+2}$ -restriction and a $PG(2, q)$ -restriction, then M has a U_{2,q^2+1} -minor.*

Proof. Suppose that M is a minimum-rank counterexample. Let $L, P \subseteq E(M)$ such that $M|L = U_{2,q+2}$ and $M|P = PG(2, q)$. If M has rank 3, then we may assume that $E(M) = P \cup \{e\}$. Since $M|P$ is modular, e is in at most one long line of M . Then, since $|P| = q^2 + q + 1$, we have $\epsilon(M/e) \geq q^2 + 1$ and, hence, M has a U_{2,q^2+1} -minor. This contradiction implies that $r(M) > 3$. Since M is round, there is an element e that is spanned by neither L nor P . Now M/e is round and contains both $M|L$ and $M|P$ as restrictions. This contradicts our choice of M . \square

The base case of the following lemma is essentially proved in [3, Lemma 2.4].

Lemma 2.3. *Let $\lambda \in \mathbb{R}$. Let k and $l \geq q \geq 2$ be positive integers, and let A and B be disjoint sets of elements in a matroid $M \in \mathcal{U}(l)$ with $\square_M(A, B) \leq k$ and $\epsilon_M(A) > \lambda q^{r_M(A)}$. Then there is a set $A' \subseteq A$ that is skew to B and satisfies $\epsilon_M(A') > \lambda l^{-k} q^{r_M(A')}$.*

Proof. By possibly contracting some elements in $B - \text{cl}_M(A)$, we may assume that A spans B and thus that $r_M(B) = \square_M(A, B)$. When $k = 1$, this means B has rank 1. We resolve this base case first.

Let e be a non-loop element of B . We may assume that A is minimal with $\epsilon_M(A) > \lambda q^{r_M(A)}$, and that $E(M) = A \cup \{e\}$. Let W be a flat of M not containing e , such that $r_M(W) = r(M) - 2$. Let H_0, H_1, \dots, H_m be the hyperplanes of M containing W , with $e \in H_0$. The sets $\{H_i - W : 1 \leq i \leq m\}$ are a disjoint cover of $E(M) - W$. Additionally, the matroid $\text{si}(M/W)$ is isomorphic to the line $U_{2,m+1}$, so we know that $m \leq l$.

By the minimality of A , we get $\epsilon_M(H_0 \cap A) \leq \lambda q^{r(M)-1}$, so

$$\epsilon_M(A - H_0) > \lambda(q - 1)q^{r(M)-1}.$$

Since the hyperplanes H_1, \dots, H_m cover $E(M) - H_0$, a majority argument gives some $1 \leq i \leq m$ such that

$$\epsilon_M(H_i \cap A) \geq \frac{1}{m} \epsilon_M(A - H_0) > \frac{\lambda}{l} (q - 1) q^{r(M)-1}.$$

Setting $A' = A \cap H_i$ gives a set of the required number of points that is skew to e and therefore to B , which is what we want.

Now suppose that the result holds for $k = t$ and consider the case that $k = t + 1$. Let A and B be disjoint sets of elements in a matroid

M with $\sqcap_M(A, B) \leq t+1$ and $\epsilon_M(A) > \lambda q^{r_M(A)}$. As mentioned earlier, we have $r_M(B) = \sqcap_M(A, B) \leq t+1$. Let e be any non-loop element of B . By the base case, there exists $A' \subseteq A$ that is skew to $\{e\}$ and satisfies $\epsilon_M(A') > \lambda l^{-1} q^{r_M(A')}$. Since $e \notin \text{cl}_M(A')$ and $r_M(B) \leq t+1$, we have $\sqcap_M(A', B) \leq t$. Now the result follows routinely by the induction hypothesis. \square

The following two results are used in the reduction of Theorem 1.2 to Theorem 1.3.

Lemma 2.4. *Let $f(k)$ be an integer-valued function such that $f(k) \geq 2f(k-1) - 1$ for each $k \geq 1$ and $f(1) \geq 1$. If M is a matroid with $\epsilon(M) \geq f(r(M))$ and $r(M) \geq 1$, then there is a round restriction N of M such that $\epsilon(N) \geq f(r(N))$ and $r(N) \geq 1$.*

Proof. We may assume that M is not round and, hence, there is a partition (A, B) of $E(M)$ such that $r_M(A) < r(M)$ and $r_M(B) < r(M)$. Clearly $r_M(A) \geq 1$ and $r_M(B) \geq 1$. Inductively we may assume that $\epsilon_M(A) < f(r_M(A))$ and $\epsilon_M(B) < f(r_M(B))$. Thus $\epsilon(M) \leq \epsilon(M|A) + \epsilon(M|B) \leq f(r_M(A)) + f(r_M(B)) - 2 \leq 2f(r(M) - 1) - 2 < f(r(M))$, which is a contradiction. \square

Lemma 2.5. *Let $q \geq 4$ and $t \geq 1$ be integers and let M be a matroid with $\epsilon(M) \geq \frac{q^{r(M)} - 1}{q - 1}$ and $r(M) \geq 3t$. If M is not round, then either M has a U_{2, q^2+2} -minor or there is a round restriction N of M such that $r(N) \geq t$ and $\epsilon(N) > \frac{q^{r(N)} - 1}{q - 1}$.*

Proof. Let $s = r(M)$ and let $f(k) = \left(\frac{q}{2}\right)^{s-k} \left(\frac{q^k - 1}{q - 1}\right)$. For any $k \geq 1$,

$$\begin{aligned} f(k+1) &= \left(\frac{q}{2}\right)^{s-k-1} \left(\frac{q^{k+1} - 1}{q - 1}\right) \\ &> \left(\frac{q}{2}\right)^{s-k-1} \left(q \frac{q^k - 1}{q - 1}\right) \\ &= 2f(k). \end{aligned}$$

Moreover $f(1) \geq 1$ and $\epsilon(M) \geq f(r(M))$. Then, by Lemma 2.4, there is a round restriction N of M such that $r(N) \geq 1$ and $\epsilon(N) \geq f(r(N))$. Since M is not round, $r(N) < r(M) = s$ and, hence, $\epsilon(N) > \frac{q^{r(N)} - 1}{q - 1}$.

We may assume that $r(N) < t$. Therefore, since $s \geq 3t$ and $q \geq 4$,

$$\begin{aligned}
\epsilon(N) &\geq f(r(N)) \\
&= \left(\frac{q}{2}\right)^{s-r(N)} \left(\frac{q^{r(N)} - 1}{q - 1}\right) \\
&\geq \left(\frac{q}{2}\right)^{2t} \left(\frac{q^{r(N)} - 1}{q - 1}\right) \\
&\geq q^t \left(\frac{q^{r(N)} - 1}{q - 1}\right) \\
&> q^{r(N)} \left(\frac{q^{r(N)} - 1}{q - 1}\right) \\
&\geq \left(\frac{q^{r(N)} + 1}{q + 1}\right) \left(\frac{q^{r(N)} - 1}{q - 1}\right) \\
&= \left(\frac{q^{2r(N)} - 1}{q^2 - 1}\right).
\end{aligned}$$

Therefore, by Theorem 1.1, M has a U_{2,q^2+2} -minor, as required. \square

3. THE MAIN RESULTS

We start with a proof of Theorem 1.4, which we restate here.

Theorem 3.1. *There is an integer-valued function $n(q)$ such that, for each prime power q , if M is a round matroid that contains a $U_{2,q+2}$ -restriction and a $PG(n(q) - 1, q)$ -minor, then M has a U_{2,q^2+1} -minor.*

Proof. Recall that the function $\alpha(l, q, n)$ was defined in Theorem 2.1. Let q be a prime power, let $\alpha = \alpha(q^2 - 1, q - 1, 3)$. Let n be an integer that is sufficiently large so that $\left(\frac{q}{q-1}\right)^n > \alpha q^5 (q - 1)^2$. We define $n(q) = n$. Suppose that the result fails for this choice of $n(q)$ and let M be a minimum-rank counterexample. Thus M is a round matroid having a line L , with at least $q + 2$ points, and a minor N isomorphic to $PG(n - 1, q)$, but $M \in \mathcal{U}(q^2 - 1)$.

Suppose that $N = M/C \setminus D$ where C is independent. If $e \in C - L$, then M/e is round, contains the line L , and has N as minor — contrary to our choice of M . Therefore $C \subseteq L$ and, hence, $r(M) \leq r(N) + 2 \leq n + 2$.

Let $X = E(M) - L$. By our choice of n , we have $\epsilon(M|(X - D)) \geq \frac{q^n - 1}{q - 1} - (q^2 + 1) = q^3 \frac{q^{n-3} - 1}{q - 1} + q \geq q^{n-1} > q^4 \alpha (q - 1)^{n+2} \geq q^4 \alpha (q - 1)^{r_M(X)}$. By Lemma 2.3, there is a flat $F \subseteq X - D$ of M that is skew to L and satisfies $\epsilon(M|F) \geq \alpha (q - 1)^{r_M(F)}$. Since F is skew to L , F is also skew

to C . Therefore $M|F = N|F$ and hence $M|F$ is $\text{GF}(q)$ -representable. Then, by Theorem 2.1, $M|F$ has a $\text{PG}(2, q)$ -minor. Therefore there is a set $Y \subseteq F$ such that $(M|F)/Y$ contains a $\text{PG}(2, q)$ -restriction. Now M/Y is round, contains a $(q+2)$ -point line, and contains a $\text{PG}(2, q)$ -restriction. Then, by Lemma 2.2, M has a U_{2, q^2+1} -minor. \square

Now we will prove Theorem 1.3 which we reformulate here. The function $n(q)$ was defined in Theorem 3.1.

Theorem 3.2. *For each prime power q , if M is a round matroid with a $\text{PG}(n(q) - 1, q)$ -minor but no U_{2, q^2+1} -minor, then $\epsilon(M) \leq \frac{q^{r(M)} - 1}{q - 1}$.*

Proof. Let M be a minimum-rank counterexample. By Lemma 2.2, $r(M) > n(q)$. Let $e \in E(M)$ be a non-loop element such that M/e has a $\text{PG}(n - 1, q)$ -minor. Note that M/e is round. Then, by the minimality of M , $\epsilon(M/e) \leq \frac{q^{r(M)-1} - 1}{q - 1}$. By Theorem 3.1, each line of M containing e has at most $q + 1$ points. Hence $\epsilon(M) \leq 1 + q\epsilon(M/e) \leq 1 + q \left(\frac{q^{r(M)-1} - 1}{q - 1} \right) = \frac{q^{r(M)} - 1}{q - 1}$. This contradiction completes the proof. \square

We can now prove our main result, Theorem 1.2, which we restate below.

Theorem 3.3. *Let $l \geq 2$ be a positive integer and let q be the largest prime power less than or equal to l . If M is a matroid with no $U_{2, l+2}$ -minor and with sufficiently large rank, then $\epsilon(M) \leq \frac{q^{r(M)} - 1}{q - 1}$.*

Proof of Theorem 1.2. When l is a prime-power, the result follows from Theorem 1.1. Therefore we may assume that $l \geq 6$ and, hence, $q \geq 5$. Recall that $n(q)$ is defined in Theorem 3.1 and $\alpha(l, q - 1, n)$ is defined in Theorem 2.1. Let $n = n(q)$ and let k be an integer that is sufficiently large so that $\left(\frac{q}{q-1} \right)^k \geq q\alpha(l, q - 1, n)$. Thus, for any $k' \geq k$, we get $\frac{q^{k'} - 1}{q - 1} \geq q^{k'-1} \geq \alpha(l, q - 1, n)(q - 1)^{k'}$. Let $M \in \mathcal{U}(l)$ be a matroid of rank at least $3k$ such that $\epsilon(M) > \frac{q^{r(M)} - 1}{q - 1}$. By Lemma 2.5, M has a round restriction N such that we have $r(N) \geq k$ and $\epsilon(N) > \frac{q^{r(N)} - 1}{q - 1} \geq \alpha(l, q - 1, n)(q - 1)^{r(N)}$. By Theorem 2.1, N has a $\text{PG}(n(q) - 1, q')$ -minor for some $q' > q - 1$. If $q' > q$, then $q' + 1 \geq l + 2$, so this projective geometry has a $U_{2, l+2}$ -minor, contradicting our hypothesis. We may therefore conclude that $q' = q$, so N has a $\text{PG}(n(q) - 1, q)$ -minor. Now we get a contradiction by Theorem 3.2. \square

4. EXTREMAL MATROIDS

In this section, we prove that the extremal matroids of large rank for Theorem 1.2 are projective geometries. We need the following result to recognize projective geometries; see Oxley [6, Theorem 6.1.1].

Lemma 4.1. *Let M be a simple matroid of rank $n \geq 4$ such that every line of M contains at least three points and each pair of disjoint lines of M is skew. Then M is isomorphic to $\text{PG}(n-1, q)$ for some prime power q .*

We can now prove our extremal characterization.

Corollary 4.2. *Let $l \geq 2$ be a positive integer and let q be the largest prime power less than or equal to l . If M is a simple matroid with no $U_{2,l+2}$ -minor, with $\epsilon(M) = \frac{q^{r(M)}-1}{q-1}$, and with sufficiently large rank, then M is a projective geometry over $\text{GF}(q)$.*

Proof. Kung [5] proved the result for the case that l is a prime-power. Therefore we may assume that $l \geq 6$ and, hence, $q \geq 5$. By Theorem 1.2, there is an integer k_1 such that, if M is a matroid with no $U_{2,l+2}$ -minor and with $r(M) \geq k_1$, then $\epsilon(M) \leq \frac{q^{r(M)}-1}{q-1}$. Recall that $n(q)$ is defined in Theorem 3.1 and $\alpha(l, q, n)$ is defined in Theorem 2.1. Let k_2 be large enough so that $\left(\frac{q}{q-1}\right)^{k_2} \geq q\alpha(l, q-1, n(q)+2)$, and $k = \max(k_1, k_2)$.

Let $M \in \mathcal{U}(l)$ be a simple matroid of rank at least $3k$ such that $\epsilon(M) = \frac{q^{r(M)}-1}{q-1}$. If M is not round, then, by Lemma 2.5, M has a round restriction N such that $r(N) \geq k$ and $\epsilon(N) > \frac{q^{r(N)}-1}{q-1}$, contrary to Theorem 1.2. Hence M is round.

From the definition of k_2 , we get $\epsilon(M) \geq \alpha(l, q-1, n(q)+2)(q-1)^{r(M)}$, so by Theorem 2.1, M has a $\text{PG}(n(q)+1, q)$ -minor. Therefore, by Theorem 3.1, each line in M has at most $q+1$ points. Consider any element $e \in E(M)$. By Theorem 1.2, $\epsilon(M/e) \leq \frac{q^{r(M)-1}-1}{q-1}$. Then

$$\begin{aligned} \epsilon(M) &\leq 1 + q\epsilon(M/e) \\ &\leq 1 + q \left(\frac{q^{r(M)-1}-1}{q-1} \right) \\ &= \frac{q^{r(M)}-1}{q-1} \\ &= \epsilon(M). \end{aligned}$$

The inequalities above must hold with equality. Therefore each line in M has exactly $q+1$ points.

If M is not a projective geometry, then, by Lemma 4.1, there are two disjoint lines L_1 and L_2 in M such that $\sqcap_M(L_1, L_2) = 1$. Let $e \in L_1$. Then L_2 spans a line with at least $q + 2$ points in M/e . Since M has a $\text{PG}(n(q) + 1, q)$ -minor, M/e contains a $\text{PG}(n(q) - 1, q)$ -minor; see [1, Lemma 5.2]. This contradicts Theorem 3.1. \square

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REFERENCES

- [1] J. Geelen, B. Gerards, G. Whittle, On Rota's conjecture and excluded minors containing large projective geometries, *J. Combin. Theory Ser. B* 96 (2006), 405-425.
- [2] J.E. Bonin, J.P.S. Kung, The number of points in a combinatorial geometry with no 8-point-line minors, *Mathematical essays in honor of Gian-Carlo Rota*, Cambridge, MA (1996), 271-284, *Progr. Math.*, 161, Birkhäuser Boston, Boston, MA, (1998).
- [3] J. Geelen, K. Kabell, Projective geometries in dense matroids, *J. Combin. Theory Ser. B* 99 (2009), 1-8.
- [4] J. Geelen, J.P.S. Kung, G. Whittle, Growth rates of minor-closed classes of matroids, *J. Combin. Theory Ser. B* 99 (2009), 420-427.
- [5] J.P.S. Kung, Extremal matroid theory, in: *Graph Structure Theory* (Seattle WA, 1991), *Contemporary Mathematics*, 147, American Mathematical Society, Providence RI, 1993, pp. 21-61.
- [6] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

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